Virtual Polywell

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The idea of the "virtual polywell" came up on talk-polywell.org recently so I decided to try to create a very simple model in that vein. The essence of a polywell fusor is a set of coils in a vacuum chamber which are held at high enough potential along with a high current to help confine electrons and ions to help bring about fusion reactions in the center. The purpose of this report is to describe a fluid model which might be used to determine various parameters to help with the construction of a real polwell fusor.

The first step in the model is the creation of the magnet coils. Since this is a virtual device, the size of the coils is taken to be physcially zero - clearly unreal, but simple to compute. There are only a few configurations which lead to a uniform magnetic field: four, six or 12 coils is all that will work. Six coils is the typical polywell configuration, so this model will be based on uniform size and spacings around the x, y and z axis.

The fundamental formulas for magnetic fields from coils can be found in text books like "Classical Electrodynamics", J.D. Jackson, Wiley and "Electromagnetic Fields and Waves", P. Lorrain and D. Corson, Freeman. The introduction of scale factors by use of dimensionless constants makes plotting more useful. A dimensionless magnetic field is:

$$\vec{b} = \frac{4\pi L}{\mu_0 I_0}\vec{B}$$

where L is the distance from the center of the fusor to the center of any coil, I_0 is the amp-turns in a coil, μ_0 is the permiability of free space and $\vec{B}$ is the magnetic field (normally in units of Tesla). I also use dimensionless position vectors:

$$\vec{u} = \frac{\vec{x}}{L}$$

Assuming the coils are pure circles we have the following for the field components of any arbitrary coil:

$$\begin{aligned}
b_x &= \int \frac{j_y r_z - j_z r_y}{\left(r_x^2 + r_y^2 + r_z^2\right)^{3/2}} du_x' du_y' du_z' \\
b_y &= \int \frac{j_z r_x - j_x r_z}{\left(r_x^2 + r_y^2 + r_z^2\right)^{3/2}} du_x' du_y' du_z' \\
b_z &= \int \frac{j_x r_y - j_y r_x}{\left(r_x^2 + r_y^2 + r_z^2\right)^{3/2}} du_x' du_y' du_z'
\end{aligned}$$

where u_k is position in the volume, u_k' is position on the coils, $r_k = u_k - u_k'$ and j_l is a dimensionless current value. Since there are six coils and three field values for each coil, there are 18 integrals for every point in the volume of interest. The coils are symmetric so only 1/8th of the volume actually needs to be computed (less if one is careful). The integrals are listed below, with superscripts indicating the coil number. Coils 1 and 2 are on the x axis, coils 3 and 4 on the y axis and coils 5 and 6 are on the z axis.

The radius of each coil is taken as RL.

$$b_x^1 = \int_0^{2\pi} \frac{\sin\varphi\,(u_z - R) + \cos\varphi\,(u_y - R\cos\varphi)}{\left[(u_x - 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_x^2 = \int_0^{2\pi} \frac{-\sin\varphi\,(u_z - R) - \cos\varphi\,(u_y - R\cos\varphi)}{\left[(u_x + 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_x^3 = \int_0^{2\pi} \frac{\cos\varphi\,(u_y - 1)}{\left[(u_x + R\cos\varphi)^2 + (u_y - 1)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_x^4 = \int_0^{2\pi} \frac{-\cos\varphi\,(u_y + 1)}{\left[(u_x + R\cos\varphi)^2 + (u_y + 1)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_x^5 = \int_0^{2\pi} \frac{-\cos\varphi\,(u_z - 1)}{\left[(u_x - R\cos\varphi)^2 + (u_y - R\cos\varphi)^2 + (u_z - 1)^2\right]^{3/2}} R\,d\varphi$$

$$b_x^6 = \int_0^{2\pi} \frac{\cos\varphi\,(u_z + 1)}{\left[(u_x - R\cos\varphi)^2 + (u_y - R\cos\varphi)^2 + (u_z + 1)^2\right]^{3/2}} R\,d\varphi$$

$$b_y^1 = \int_0^{2\pi} \frac{-\cos\varphi\,(u_x - 1)}{\left[(u_x - 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_y^2 = \int_0^{2\pi} \frac{\cos\varphi\,(u_x + 1)}{\left[(u_x + 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_y^3 = \int_0^{2\pi} \frac{-\cos\varphi\,(u_x + R\cos\varphi) + \sin\varphi(u_z - R\sin\varphi)}{\left[(u_x + R\cos\varphi)^2 + (u_y - 1)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_y^4 = \int_0^{2\pi} \frac{\cos\varphi\,(u_x + R\cos\varphi) - \sin\varphi(u_z - R\sin\varphi)}{\left[(u_x + R\cos\varphi)^2 + (u_y + 1)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_y^5 = \int_0^{2\pi} \frac{-\sin\varphi\,(u_z - 1)}{\left[(u_x - R\cos\varphi)^2 + (u_y - R\cos\varphi)^2 + (u_z - 1)^2\right]^{3/2}} R\,d\varphi$$

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$$b_z^1 = \int_0^{2\pi} \frac{-\sin\varphi\,(u_x - 1)}{\left[(u_x - 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_z^2 = \int_0^{2\pi} \frac{\sin\varphi\,(u_x + 1)}{\left[(u_x + 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_z^3 = \int_0^{2\pi} \frac{-\sin\varphi\,(u_y - 1)}{\left[(u_x + R\cos\varphi)^2 + (u_y - 1)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_z^4 = \int_0^{2\pi} \frac{\sin\varphi(u_y + 1)}{\left[(u_x + R\cos\varphi)^2 + (u_y + 1)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi$$

$$b_z^5 = \int_0^{2\pi} \frac{\sin\varphi\,(u_y - R\sin\varphi) + \cos\varphi(u_x - R\cos\varphi)}{\left[(u_x - R\cos\varphi)^2 + (u_y - R\cos\varphi)^2 + (u_z - 1)^2\right]^{3/2}} R\,d\varphi$$

Each coil is also set to a high voltage. The entire system is placed inside a spherical vacuum chamber which is grounded. We can easily find the electric field from any single point charge inside a grounded sphere using the method of images. Then by principle of superposition, we can sum over all the charges on a coil. This again gives rise to 18 integrals.

Let a be the radius of the grounded sphere and b be the distance from the center to the charge in question, then the potential for a point inside a sphere is given by

$$\Phi(\vec{u}) = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{\left(r_x^2+r_y^2+r_z^2\right)^{1/2}} - \frac{\frac{a}{b}}{\left[\left(u_x-\frac{a^2}{b^2}u_x'\right)^2+\left(u_y-\frac{a^2}{b^2}u_y'\right)^2+\left(u_z-\frac{a^2}{b^2}u_z'\right)^2\right]}\right\}$$

If the charge is a differential element on the coil, we can take $dq = 4\pi\varepsilon_0 V dl$ where dl is an element of length along the coil. Since we seek a dimensionless representation of the electric field, I divide out the V and since electric field is in terms of volts/meter, I multiply through by L, then compute the gradient of the result relative to dimensionless u_j.

The general formulas for the electric field are found by taking the gradient of Φ with the knowledge that a is the radius of the ground sphere in units of L, b is distance from center of the fusor to the coil and is easily found to be $b = \sqrt{R^2+1}$ also in units of L. I find

$$\mathcal{E}_x = \int_0^{2\pi}\left\{\frac{(u_x-u_x')}{\left[(u_x-u_x')^2+\left(u_y-u_y'\right)^2+(u_z-u_z')^2\right]^{3/2}}\right.$$

$$\left. - \frac{\frac{a}{b}\left(u_x-\frac{a^2}{b^2}u_x'\right)}{\left[\left(u_x-\frac{a^2}{b^2}u_x'\right)^2+\left(u_y-\frac{a^2}{b^2}u_y'\right)^2+\left(u_z-\frac{a^2}{b^2}u_z'\right)^2\right]^{3/2}}\right\} R\,d\varphi$$

$$\mathcal{E}_y = \int_0^{2\pi}\left\{\frac{(u_y-u_y')}{\left[(u_x-u_x')^2+\left(u_y-u_y'\right)^2+(u_z-u_z')^2\right]^{3/2}}\right.$$

$$\left. - \frac{\frac{a}{b}\left(u_y-\frac{a^2}{b^2}u_y'\right)}{\left[\left(u_x-\frac{a^2}{b^2}u_x'\right)^2+\left(u_y-\frac{a^2}{b^2}u_y'\right)^2+\left(u_z-\frac{a^2}{b^2}u_z'\right)^2\right]^{3/2}}\right\} R\,d\varphi$$

$$\mathcal{E}_z = \int_0^{2\pi}\left\{\frac{(u_z-u_z')}{\left[(u_x-u_x')^2+\left(u_y-u_y'\right)^2+(u_z-u_z')^2\right]^{3/2}}\right.$$

$$\left. - \frac{\frac{a}{b}\left(u_z-\frac{a^2}{b^2}u_z'\right)}{\left[\left(u_x-\frac{a^2}{b^2}u_x'\right)^2+\left(u_y-\frac{a^2}{b^2}u_y'\right)^2+\left(u_z-\frac{a^2}{b^2}u_z'\right)^2\right]^{3/2}}\right\} R\,d\varphi$$

Once I had computed the magnetic fields using the above 18 equations, I modified the code to use the same 18 subroutines with similar denominators. The parameter $\frac{a}{b}$ was passed as an argument and both terms at the same $d\varphi$ step were added to the integral. The plots of these dimensionless electric and magnetic fields are seen below:

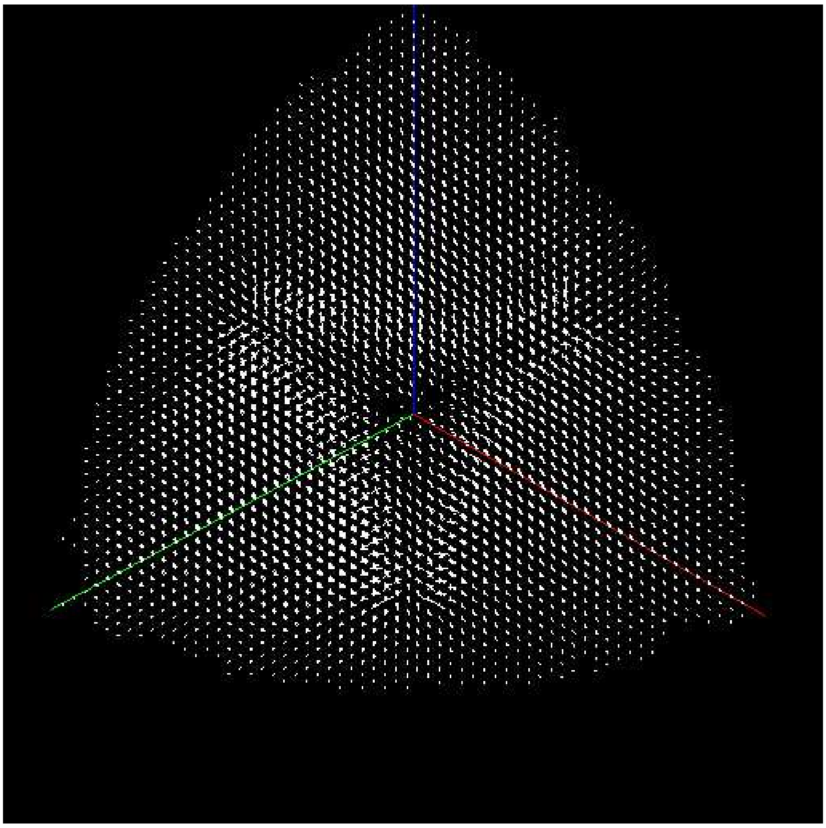

Figure 1. Electric field

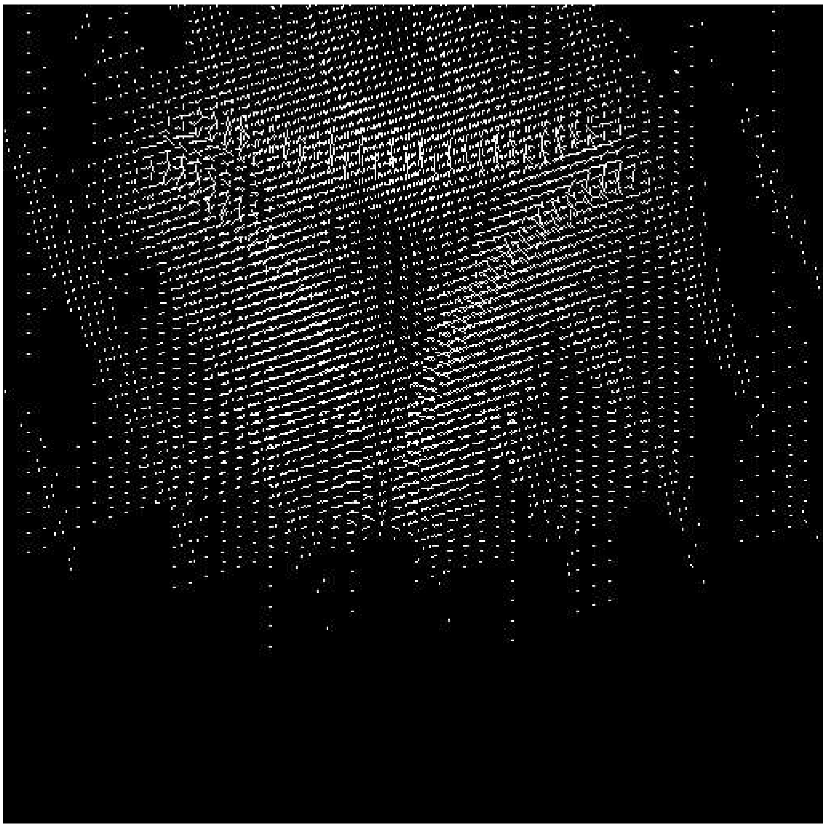

Figure 2. Magnetic field

To approach the physics of the polywell fusor one has a lot of approximations to choose from. I choose to use the basic plasma physics fluid model because it has a lot of history in electric and magnetic field environments which shows how useful the model can be. The basic equations are taken from "Principles of Plasma Physics", N.A. Krall and A.W. Trivelpiece, McGraw-Hill. In SI units we have

$$\frac{\partial f_e}{\partial t} + \vec{v} \cdot \vec{\nabla} f_e - \frac{e}{m}\left(\vec{E}_T + \vec{v} \times \vec{B}_T\right) \cdot \vec{\nabla}_v f_e = \frac{\partial f_e}{\partial t}|_c$$

$$\vec{\nabla} \cdot \vec{E}_T = -\frac{n_e e}{\varepsilon_0} \int f_e d\vec{v} + \frac{\rho_{\text{ext}}}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{B}_T = \mu_0 \varepsilon_0 \frac{\partial \vec{E}_T}{\partial t} - \mu_0 n_e e \int \vec{v} f_e d\vec{v} + \mu_0 \vec{J}_{\text{ext}}$$

$$\vec{\nabla} \times \vec{E}_T = -\frac{\partial \vec{B}_T}{\partial t}$$

Here, f_e is the electron distribution function, $\vec{v}$ is the velocity vector, e is charge and m is the mass of an electron, $\vec{\nabla}_v$ is gradient with respect to velocity, n_e is the average electron density, ρ_{ext} is external electron distribution (not moving), $\vec{J}_{\text{ext}}$ is the moving external current density and $|_c$ refers to collisions. This last term will be ignored from here on out, but it should be noted this is where collisions enter the picture.

To make use of the external fields described above we can easily separate the total electric and magnetic fields into several components. The external fields are given by

$$\vec{\nabla} \cdot \vec{E}_{\text{ext}} = \frac{\rho_{\text{ext}}}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{E}_{\text{ext}} = -\frac{\partial \vec{B}_{\text{ext}}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B}_{\text{ext}} = 0$$

$$\vec{\nabla} \times \vec{B}_{\text{ext}} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}_{\text{ext}}}{\partial t} + \mu_0 \vec{J}_{\text{ext}}$$

To start with, we take the fields in the coils as steady state and voltage on them as steady state as well so there is no electromagnetic coupling. The POPS design can include these terms later, and by writing everything fully it is easy to see where to put the fluctuating fields back in.

Using $\vec{E}$ and $\vec{B}$ as the fields created from the electrons it is easy to find the equations that show the connections between the electron fluid distribution and forces acting on it. subtracting the external equations from the particle distribution equations gives

$$\vec{\nabla} \cdot \vec{E} = -\frac{n_e e}{\varepsilon_0} \int f_e(\vec{x}, \vec{v}, t) d\vec{v}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} - \mu_0 n_e e \int \vec{v} f_e(\vec{x}, \vec{v}, t) d\vec{v}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

If we assume (for the moment) that the fields created by the electrons is much weaker than the fields created by the coils, then we can make some very crude estimates of the behavior of the fluid. While it is obviously inaccurate for a real fusor, it does give some ideas on where to mount electron guns for injection and where *not* to mount electron guns as well.

With the weak field assumption the particle motion equation becomes

$$\frac{\partial f_e}{\partial t} + \vec{v} \cdot \vec{\nabla} f_e - \frac{e}{m}\left(\vec{E}_{\text{ext}} + \vec{v} \times \vec{B}_{\text{ext}}\right) \cdot \vec{\nabla}_v f_e = 0$$

What I'd like to do now is transform this from a unit equation to a unitless equation to make a clear connection between the computed fields for the electrostatic and magnetostatic external fields already described. The units on the distrbution function are length^{-3} times velocity^{-3}. The units on $\vec{v}$ is velocity, $\vec{E}_{\text{ext}}$ is Volt/length, and the magnetic field is described in the first equation in this article (Tesla in SI units). To make this formula dimensionless, I multiply by (length^3 times velocity^3) along with writing the electric and magnetic fields in their dimensionless forms. I use f for the dimensionless particle distribution function, $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ for the dimensionless electric and magnetic fields, $\vec{\mathfrak{u}}$ for dimensionless velocity and $\vec{\mathfrak{r}}$ for dimensionless postion. The result of all this machination is

$$\frac{\partial f}{\partial t} = -\frac{v_0}{L} \vec{\mathfrak{u}} \cdot \frac{\partial f}{\partial \vec{\mathfrak{r}}} + \left(\frac{eV}{mLv_0} \vec{\mathcal{E}} + \frac{e\mu_0 I_0}{4\pi m L} \vec{\mathfrak{u}} \times \vec{\mathcal{B}}\right) \cdot \frac{\partial f}{\partial \vec{\mathfrak{u}}}$$

where v_0 is some arbitrary velocity which makes the terms dimensionless (inverse time actually, but I'll fix that in a minute). First I want to define the arbitrary velocity in terms of other variables which the problem has control over, then I'll work on moving terms around to a more convienient description.

The most obvious choice for a fundamental velocity is to assume that an electron which accelerates from dead still to the voltage on the grid has converted all its potential energy to kinetic energy. This gives

$$v_0 = \sqrt{\frac{2eV}{m}}$$

Putting this in for the arbitrary velocity and moving terms around we get

$$\frac{4\pi m L}{e\mu_0 I_0}\frac{\partial f}{\partial t} = -2\mathcal{C}_p\vec{u}\cdot\frac{\partial f}{\partial \vec{r}} + \left(\mathcal{C}_p\vec{\mathcal{E}} + \vec{u}\times\vec{\mathcal{B}}\right)\cdot\frac{\partial f}{\partial \vec{u}}$$

where

$$\mathcal{C}_p = \frac{2\pi}{\mu_0 I_0}\sqrt{\frac{2mV}{e}}$$

is what I call the "confinement parameter". This parameter is dimensionless so the whole equation can be scaled independent of real world constraints. Understanding the fluid distribution this way allows us to find reasonable combinations of voltages and currents which will create a workable fusion device, possibly of different sizes. It also helps to point out obvious combinations which simply can not work.

It is interesting that the time scale which makes the equation dimensionless is independent of the grid voltage. The time scale is given by the coefficient of the left hand side:

$$\text{time} = \frac{4\pi m L}{e\mu_0 I_0}$$

The Larmor frequency of an electron in a magnetic field is given by

$$\omega = \frac{eB}{m}$$

where B is the magnetic field in Tesla. Comparing these two equations we see that the fundamental magnetic field strength can be related to the Larmor frequency if we take

$$B = \frac{\mu_0 I_0}{2L}$$

The dimension of the device thus give us a fundamental scale for the Larmor frequency along with the current in the coil and the voltage on the grid. Plots of $\mathcal{C}_p\vec{\mathcal{E}} + \vec{u}\times\vec{\mathcal{B}}$ for several values of $\mathcal{C}_p$ give us an idea of what the forces are on an electron fluid at various points in the polywell. Since this is a multidimensional space (3 dimensions for space, 3 for velocity and 1 (or two) for $\mathcal{C}_p$) it is non-trivial to get a feel for what is going on inside a polywell.

It should be clear that computing electron distributions over time is straight forward. Some assumptions on where to inject electrons in the first place need to be made using real physical devices, but a study of the "force volume" $\mathcal{C}_p\vec{\mathcal{E}} + \vec{u}\times\vec{\mathcal{B}}$ will help to define the ideal electron gun locations.

It is clear that even simple models are fairly complicated. Only building a polywell device will tell us the real story. Models can help us find the best bet on what will work, and can certainly tell us what to avoid.