“Wire Coil” Magnetic Field for Polywell

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Recently on Talk-Polywell.org one member named “tombo” suggested a very interesting coil layout based on triangles. Since this is simply a sequence of lines which are chords cutting between points on the surface of a sphere, it seemed like this should be theoretically easy to compute. The following notes describe the theory and point the way towards code which I hope to write over the next week or so.

Let’s start with a general description of a line in 3D space. If we have two points (x_1, y_1, z_1) and (x_2, y_2, z_2) then we can mark a line between them using parametric form:

$$x = x_1 + t(x_2 - x_1) \tag{1}$$

$$y = y_1 + t(y_2 - y_1) \tag{2}$$

$$z = z_1 + t(z_2 - z_1) \tag{3}$$

where t runs between 0 and 1 to range from the first to second point. For any line segment, we can take

$$a = (x_2 - x_1), b = (y_2 - y_1), c = (z_2 - z_1) \tag{4}$$

to be constants of the line.

For current flowing from point 1 to point 2 we can take the current direction as

$$\vec{J} = \frac{a\hat{x} + b\hat{y} + c\hat{z}}{l} I_0 \tag{5}$$

where $l = \sqrt{a^2 + b^2 + c^2}$ and I_0 being the current magnitude.

The magnetic field from a line current is defined as (see “Electromagnetic Fields and Waves”, P. Lorrain & D. Corson, Freeman, 1970 for examples)

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{r}}{r^3} dl \tag{6}$$

where $\vec{r}$ is the vector from a point on the line current to the field point in question:

$$\vec{r} = \vec{r_r} - \vec{r}_t = (x_r - x_1 - at)\hat{x} + (y_r - y_1 - bt)\hat{y} + (z_r - z_1 - ct)\hat{z} \tag{7}$$

When we take the cross product of (7) with (5), the parallel terms will cancel (the factors multiplied by t in (7)). This makes the integral straight forward. The integration along the wire is given by

$$dl = \sqrt{a^2 + b^2 + c^2} dt = l dt \tag{8}$$

In addition define the following:

$$x' = x_r - x_1 \tag{9}$$

$$y' = y_r - y_1 \tag{10}$$

$$z' = z_r - z_1 \tag{11}$$

Combining all the above into one formula I get

$$\vec{B} = \frac{\mu_0 I_0}{4\pi}\left[(b\,z' - c\,y')\hat{x} + (c\,x' - a\,z')\hat{y} + (a\,y' - bx')\hat{z}\right]\int_0^1 \frac{dt}{[(x_1 - x_r + at)^2 + (y_1 - y_r + bt)^2 + (z_1 - z_r + ct)^2]^{\frac{3}{2}}} \tag{12}$$

This integral is analytic, so can be evaluated symbolicly. We make the following substitutions:

$$r'^2 = (x_1 - x_r)^2 + (y_1 - y_r)^2 + (z_1 - z_r)^2 = x'^2 + y'^2 + z'^2 \tag{13}$$

$$k = 2[a(x_1 - x_r) + b(y_1 - y_r) + c(z_1 - z_r)] \tag{14}$$

and write the integral in equation (9) as

$$\int_0^1 \frac{dt}{\sqrt{\left(r'^2 + kt + l^2 t^2\right)^3}} \tag{15}$$

From Gradshteyn & Ryzhik 2.264.5 we find

$$\int \frac{dx}{\sqrt{(a + bx + \mathrm{cx}^2)^3}} = \frac{2(2cx + b)}{\triangle\sqrt{a + bx + \mathrm{cx}^2}} \tag{16}$$

where

$$\triangle = 4r'^2 l^2 - k^2 \tag{17}$$

It is easy to see that $\triangle > 0$ if we take $r'^2 = \vec{r}\,' \cdot \vec{r}\,'$, $k = 2\vec{l} \cdot \vec{r}\,'$, and $l^2 = \vec{l} \cdot \vec{l}$. This is what ensures (13) is always the correct answer. The only very special case that is possible is $\triangle = 0$ which happens for points on the line. We will ignore these points here.

The full field from a single wire segment is thus

$$\vec{B} = \frac{\mu_0 I_0}{2\pi}\frac{[(bz' - cy')\hat{x} + (cx' - az')\hat{y} + (ay' - bx')\hat{z}]}{4r'^2 l^2 - k^2}\left[\frac{2l^2 - k}{\sqrt{r'^2 - k + l}} + \frac{k}{r'}\right] \tag{18}$$

It is straight forward to find the end points of line segments which fit the triangular segments described in tombo's posts. Equation (15) can then be summed over all line segments for each point in space to find the complete magnetic field.