Particle Equations in Polywell

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January 16, 2008

The following is mearly a write up of one section of my notes of trying to work on the Polywell reactor physics. It is not meant to be useful, it was meant as an excercise in figuring out single particle motion in a complicated electric and magnetic field.

We start with

$$\hat{F} = \frac{d\hat{p}}{dt} = m\frac{d\vec{v}}{dt} = -e\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{1}$$

Expanding into all three dimensions gives

$$m\frac{dv_x}{dt} = -e(E_x + v_y B_z - v_z B_y) \tag{2}$$

$$m\frac{dv_y}{dt} = -e(E_y + v_z B_x - v_x B_z) \tag{3}$$

$$m\frac{dv_z}{dt} = -e(E_z + v_x B_y - v_y B_x) \tag{4}$$

In formal notation let $D' = \frac{d}{dt}$ and rewrite these as

$$\frac{m}{e}D'v_x + B_z v_y - B_y v_z = -E_x \tag{5}$$

$$-B_z v_x + \frac{m}{e}D'v_y + B_x v_z = -E_y \tag{6}$$

$$B_y v_x - B_x v_y + \frac{m}{e}D'v_z = -E_z \tag{7}$$

Transforming to my dimensionless variables, let $D = \frac{d}{d\nu}$:

$$\frac{m}{e}\frac{e\mu_0 I_0}{4\pi m L}D(v_0 u_x) + \frac{\mu_0 I_0}{4\pi L}\mathcal{B}_z v_0 u_y - \frac{\mu_0 I_0}{4\pi L}\mathcal{B}_y v_0 u_z = -\frac{V}{L}\mathcal{E}_x \tag{8}$$

Since

$$\frac{\mu_0 I_0}{4\pi L}v_0 = \frac{\mu_0 I_0}{4\pi L}\sqrt{\frac{2eV}{m}} = \frac{\mu_0 I_0 V}{2\pi L}\sqrt{\frac{e}{2mV}} = \frac{V}{LC_p} \tag{9}$$

we get

$$Du_x + \mathcal{B}_z u_y - \mathcal{B}_y u_z = -C_p\mathcal{E}_x \tag{10}$$

$$-\mathcal{B}_z u_x + Du_y + \mathcal{B}_x u_z = -C_p\mathcal{E}_y \tag{11}$$

$$\mathcal{B}_y u_x - \mathcal{B}_x u_y + Du_z = -C_p\mathcal{E}_z \tag{12}$$

To solve this for every point in space we take $\vec{\mathcal{B}}$ and $\vec{\mathcal{E}}$ as functions of space (and it could be time too, but that's getting way too hard.) Formally we can write a solution to 10-12 as

$$\Delta = \begin{vmatrix} D & \mathcal{B}_z & -\mathcal{B}_y \\ -\mathcal{B}_z & D & \mathcal{B}_x \\ \mathcal{B}_y & -\mathcal{B}_x & D \end{vmatrix} = D^3 + \mathcal{B}_x\mathcal{B}_y\mathcal{B}_z - \mathcal{B}_x\mathcal{B}_y\mathcal{B}_z + D\mathcal{B}_y^2 + D\mathcal{B}_x^2 + D\mathcal{B}_z^2 \tag{13}$$

and we get

$$\Delta = D^3 + \left(\mathcal{B}_y^2 + \mathcal{B}_x^2 + \mathcal{B}_z^2\right) D \tag{14}$$

Putting in the driving terms for the first column gives

$$\Delta u_x = \begin{vmatrix} -C_p\mathcal{E}_x & \mathcal{B}_z & -\mathcal{B}_y \\ -C_p\mathcal{E}_y & D & \mathcal{B}_x \\ -C_p\mathcal{E}_z & -\mathcal{B}_x & D \end{vmatrix} = -C_p\left[\mathcal{E}_x D^2 + \mathcal{E}_y\mathcal{B}_x\mathcal{B}_y + \mathcal{E}_z\mathcal{B}_x\mathcal{B}_z + \mathcal{E}_x\mathcal{B}_x^2 + (\mathcal{E}_z\mathcal{B}_y - \mathcal{E}_y\mathcal{B}_z)D\right] \tag{15}$$

The homogeneous solution comes from

$$D(D^2 + \mathcal{B}_y^2 + \mathcal{B}_x^2 + \mathcal{B}_z^2) = 0 \tag{16}$$

Let

$$u_h = e^{r\nu} \tag{17}$$

be a homogeneous solution. Then

$$D u_h = r e^{r\nu} \tag{18}$$

and we get

$$r(r^2 + \mathcal{B}_y^2 + \mathcal{B}_x^2 + \mathcal{B}_z^2) = 0 \tag{19}$$

So we have

$$r = 0, r = \pm i\sqrt{\mathcal{B}_y^2 + \mathcal{B}_x^2 + \mathcal{B}_z^2} \tag{20}$$

Assume

$$u_{xh} = b_k e^{r_k\nu} \tag{21}$$

$$u_{yh} = c_k e^{r_k\nu} \tag{22}$$

$$u_{zh} = d_k e^{r_k\nu} \tag{23}$$

Plugging these into 10 - 12 with no driving terms gives

$$b_k r_k + \mathcal{B}_z c_k - \mathcal{B}_y d_k = 0 \tag{24}$$

$$-\mathcal{B}_z b_k + c_k r_k + \mathcal{B}_x d_k = 0 \tag{25}$$

$$\mathcal{B}_y b_k - \mathcal{B}_x c_k + d_k r_k = 0 \tag{26}$$

From the last equation

$$c_k = \frac{\mathcal{B}_y b_k + d_k r_k}{\mathcal{B}_x} \tag{27}$$

and from the first and second equations

$$d_k = -\frac{\mathcal{B}_x r_k + \mathcal{B}_z \mathcal{B}_y}{\mathcal{B}_z r_k - \mathcal{B}_x \mathcal{B}_y} \tag{28}$$

So the homogeneous solution is complete, but the particular solution is a huge mess. A particular solution can be found by setting 14 equal to 15 using 10-12. Letting

$$u_x \propto e^{r_l \nu} \tag{29}$$

we have

$$D u_x \propto r_l e^{r_l \nu} \tag{30}$$

and setting 14 = 15 with a little moving around gives

$$r_l^3 + C_p \mathcal{E}_x r_l^2 + r_l \left[\mathcal{B}_y^2 + \mathcal{B}_x^2 + \mathcal{B}_z^2 + C_p(\mathcal{E}_z \mathcal{B}_y - \mathcal{E}_y \mathcal{B}_z)\right] + C_p \mathcal{B}_x (\mathcal{E}_x \mathcal{B}_x + \mathcal{E}_y \mathcal{B}_y + \mathcal{E}_z \mathcal{B}_z) = 0 \tag{31}$$

Let

$$s = r_l - \frac{C_p \mathcal{E}_x}{3} \tag{32}$$

Then we get

$$s^3 + as + b = 0 \tag{33}$$

where

$$a = \vec{\mathcal{B}} \cdot \vec{\mathcal{B}} + C_p(\mathcal{E}_z \mathcal{B}_y - \mathcal{E}_y \mathcal{B}_z) - \frac{C_p^2 \mathcal{E}_x^2}{3} \tag{34}$$

and

$$b = 2\frac{C_p^3 \mathcal{E}_x^3}{27} - \frac{C_p \mathcal{E}_x}{3}\left[\vec{\mathcal{B}} \cdot \vec{\mathcal{B}} + C_p(\mathcal{E}_z \mathcal{B}_y - \mathcal{E}_y \mathcal{B}_z)\right] + C_p \mathcal{B}_x \left(\vec{\mathcal{E}} \cdot \vec{\mathcal{B}}\right) \tag{35}$$

The general solution to 33 is given by (CRC Standard Mathematical Tables 21st edition)

$$s = A + B, -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3} \tag{36}$$

and

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \tag{37}$$

$$B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \tag{38}$$

Now, to save some writing attempting to solve the cubic equation 33 let

$$\alpha = \vec{\mathcal{B}} \cdot \vec{\mathcal{B}} + C_p(\mathcal{E}_z \mathcal{B}_y - \mathcal{E}_y \mathcal{B}_z) \tag{39}$$

$$\gamma = C_p \mathcal{E}_x \tag{40}$$

and

$$\delta = \frac{\mathcal{B}_x}{\mathcal{E}_x}\left(\vec{\mathcal{E}} \cdot \vec{\mathcal{B}}\right) \tag{41}$$

then 34 and 35 are

$$a = \alpha - \frac{\gamma^2}{3} \tag{42}$$

$$b = \frac{2}{27}\gamma^3 - \frac{\gamma\alpha}{3} + \gamma\delta \tag{43}$$

Grinding through a bunch of algebra I find

$$-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} = -\frac{\gamma}{54}(2\gamma^2 - 9\alpha + 27\delta) \pm \frac{1}{\sqrt{3}}\left[\left(\frac{\alpha^2}{9} - \gamma^2\delta\right)\left(\alpha - \frac{\gamma^2}{4}\right) + \frac{\gamma^2}{4}\delta\left(\frac{5}{9}\gamma^2 - 3\delta\right)\right]^{1/2} \tag{44}$$

If $\frac{b^2}{4} + \frac{a^3}{27} = 0$ then $A = B$. The solutions will be of the form

$$e^{(A+B)\nu}, e^{-\frac{A+B}{2}\nu}, te^{-\frac{A+B}{2}\nu} \tag{45}$$

If $\frac{b^2}{4} + \frac{a^3}{27} > 0$ then A and B are real and we have one exponential and two sinesoidal solutions.

If $\frac{b^2}{4} + \frac{a^3}{27} < 0$ then A and B are imaginary and we have one sinesoidal solution and two exponentials. However, the exponentials also have sinesoidal components.

At this point, I gave up on this kind of solution method. It is much easier to do brute force than attempt to grind out all the possible cases and keep track of all this crap.

Note that the subscripts in 34 and 35 rotate for the y and z dimensions - it is exactly the same math. However, at any one point in space, you may have either sinesoidal or exponential solutions in any of the dimensions. Brute force just seems really simple in comparison.