Virtual Polywell

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November 18, 2007

The idea of the "virtual polywell" came up on talk-polywell.org recently so I decided to try to create a very simple model in that vein. The essence of a polywell fusor is a set of coils in a vacuum chamber which are held at high enough potential along with a high current to help confine electrons and ions to help bring about fusion reactions in the center. The purpose of this report is to describe a fluid model which might be used to determine various parameters to help with the construction of a real polwell fusor.

The first step in the model is the creation of the magnet coils. Since this is a virtual device, the size of the coils is taken to be physcially zero - clearly unreal, but simple to compute. There are only a few configurations which lead to a uniform magnetic field: four, six or 12 coils is all that will work. Six coils is the typical polywell configuration, so this model will be based on uniform size and spacings around the x, y and z axis.

The fundamental formulas for magnetic fields from coils can be found in text books like "Classical Electrodynamics", J.D. Jackson, Wiley and "Electromagnetic Fields and Waves", P. Lorrain and D. Corson, Freeman. The introduction of scale factors by use of dimensionless constants makes plotting more useful. A dimensionless magnetic field is:

$$\vec{b} = \frac{4\pi L}{\mu_0 I_0}\vec{B} \tag{1}$$

where L is the distance from the center of the fusor to the center of any coil, I_0 is the amp-turns in a coil, μ_0 is the permeability of free space and $\vec{B}$ is the magnetic field (normally in units of Tesla). I also use dimensionless position vectors:

$$\vec{u} = \frac{\vec{x}}{L} \tag{2}$$

Assuming the coils are pure circles we have the following for the field components of any arbitrary coil:

$$\begin{aligned}
b_x &= \int \frac{j_y r_z - j_z r_y}{\left(r_x^2 + r_y^2 + r_z^2\right)^{3/2}} du'_x du'_y du'_z \\
b_y &= \int \frac{j_z r_x - j_x r_z}{\left(r_x^2 + r_y^2 + r_z^2\right)^{3/2}} du'_x du'_y du'_z \\
b_z &= \int \frac{j_x r_y - j_y r_x}{\left(r_x^2 + r_y^2 + r_z^2\right)^{3/2}} du'_x du'_y du'_z
\end{aligned} \tag{3}$$

where u_k is position in the volume, u'_k is position on the coils, $r_k = u_k - u'_k$ and j_l is a dimensionless current value. Since there are six coils and three field values for each coil, there are 18 integrals for every point in the volume of interest. The coils are symmetric so only 1/8th of the volume actually needs to be computed (less if one is careful). The integrals are listed below, with superscripts indicating the coil number. Coils 1 and 2 are on the x axis, coils 3 and 4 on the y axis and coils 5 and 6 are on the z axis.

The radius of each coil is taken as RL.

$$b_x^1 = \int_0^{2\pi} \frac{\sin\varphi\,(u_z - R) + \cos\varphi\,(u_y - R\cos\varphi)}{\left[(u_x - 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi \tag{4}$$

$$b_x^2 = \int_0^{2\pi} \frac{-\sin\varphi\,(u_z - R) - \cos\varphi\,(u_y - R\cos\varphi)}{\left[(u_x + 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi \tag{5}$$

$$b_x^3 = \int_0^{2\pi} \frac{\cos\varphi\,(u_y - 1)}{\left[(u_x + R\cos\varphi)^2 + (u_y - 1)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi \tag{6}$$

$$b_x^4 = \int_0^{2\pi} \frac{-\cos\varphi\,(u_y + 1)}{\left[(u_x + R\cos\varphi)^2 + (u_y + 1)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi \tag{7}$$

$$b_x^5 = \int_0^{2\pi} \frac{-\cos\varphi\,(u_z - 1)}{\left[(u_x - R\cos\varphi)^2 + (u_y - R\cos\varphi)^2 + (u_z - 1)^2\right]^{3/2}} R\,d\varphi \tag{8}$$

$$b_x^6 = \int_0^{2\pi} \frac{\cos\varphi\,(u_z + 1)}{\left[(u_x - R\cos\varphi)^2 + (u_y - R\cos\varphi)^2 + (u_z + 1)^2\right]^{3/2}} R\,d\varphi \tag{9}$$

$$b_y^1 = \int_0^{2\pi} \frac{-\cos\varphi\,(u_x - 1)}{\left[(u_x - 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi \tag{10}$$

$$b_y^2 = \int_0^{2\pi} \frac{\cos\varphi\,(u_x + 1)}{\left[(u_x + 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi \tag{11}$$

$$b_y^3 = \int_0^{2\pi} \frac{-\cos\varphi\,(u_x + R\cos\varphi) + \sin\varphi\,(u_z - R\sin\varphi)}{\left[(u_x + R\cos\varphi)^2 + (u_y - 1)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi \tag{12}$$

$$b_y^4 = \int_0^{2\pi} \frac{\cos\varphi\,(u_x + R\cos\varphi) - \sin\varphi\,(u_z - R\sin\varphi)}{\left[(u_x + R\cos\varphi)^2 + (u_y + 1)^2 + (u_z - R\sin\varphi)^2\right]^{3/2}} R\,d\varphi \tag{13}$$

$$b_y^5 = \int_0^{2\pi} \frac{-\sin\varphi\,(u_z - 1)}{\left[(u_x - R\cos\varphi)^2 + (u_y - R\cos\varphi)^2 + (u_z - 1)^2\right]^{3/2}} R\,d\varphi \tag{14}$$

$$b_y^6 = \int_0^{2\pi} \frac{\sin\varphi\,(u_z + 1)}{\left[(u_x - R\cos\varphi)^2 + (u_y - R\cos\varphi)^2 + (u_z + 1)^2\right]^{3/2}} R\,d\varphi \tag{15}$$

$$b_z^1 = \int_0^{2\pi} \frac{-\sin\varphi\,(u_x - 1)}{[(u_x - 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2]^{3/2}} R\,d\varphi \tag{16}$$

$$b_z^2 = \int_0^{2\pi} \frac{\sin\varphi\,(u_x + 1)}{[(u_x + 1)^2 + (u_y - R\cos\varphi)^2 + (u_z - R\sin\varphi)^2]^{3/2}} R\,d\varphi \tag{17}$$

$$b_z^3 = \int_0^{2\pi} \frac{-\sin\varphi\,(u_y - 1)}{[(u_x + R\cos\varphi)^2 + (u_y - 1)^2 + (u_z - R\sin\varphi)^2]^{3/2}} R\,d\varphi \tag{18}$$

$$b_z^4 = \int_0^{2\pi} \frac{\sin\varphi\,(u_y + 1)}{[(u_x + R\cos\varphi)^2 + (u_y + 1)^2 + (u_z - R\sin\varphi)^2]^{3/2}} R\,d\varphi \tag{19}$$

$$b_z^5 = \int_0^{2\pi} \frac{\sin\varphi\,(u_y - R\sin\varphi) + \cos\varphi\,(u_x - R\cos\varphi)}{[(u_x - R\cos\varphi)^2 + (u_y - R\cos\varphi)^2 + (u_z - 1)^2]^{3/2}} R\,d\varphi \tag{20}$$

Each coil is also set to a high voltage. The entire system is placed inside a spherical vacuum chamber which is grounded. We can easily find the electric field from any single point charge inside a grounded sphere using the method of images. Then by principle of superposition, we can sum over all the charges on a coil. This again gives rise to 18 integrals.

Let a be the radius of the grounded sphere and b be the distance from the center to the charge in question, then the potential for a point inside a sphere is given by

$$\Phi(\vec{u}) = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{\left(r_x^2+r_y^2+r_z^2\right)^{1/2}} - \frac{\frac{a}{b}}{\left[\left(u_x-\frac{a^2}{b^2}u_x'\right)^2+\left(u_y-\frac{a^2}{b^2}u_y'\right)^2+\left(u_z-\frac{a^2}{b^2}u_z'\right)^2\right]}\right\} \tag{22}$$

If the charge is a differential element on the coil, we can take $dq = 4\pi\varepsilon_0 V dl$ where dl is an element of length along the coil. Since we seek a dimensionless representation of the electric field, I divide out the V and since electric field is in terms of volts/meter, I multiply through by L, then compute the gradient of the result relative to dimensionless u_j.

The general formulas for the electric field are found by taking the gradient of Φ with the knowledge that a is the radius of the ground sphere in units of L, b is distance from center of the fusor to the coil and is easily found to be $b = \sqrt{R^2+1}$ also in units of L. I find

$$\mathcal{E}_x = \int_0^{2\pi}\left\{\frac{(u_x-u_x')}{\left[(u_x-u_x')^2+\left(u_y-u_y'\right)^2+(u_z-u_z')^2\right]^{3/2}} - \frac{\frac{a}{b}\left(u_x-\frac{a^2}{b^2}u_x'\right)}{\left[\left(u_x-\frac{a^2}{b^2}u_x'\right)^2+\left(u_y-\frac{a^2}{b^2}u_y'\right)^2+\left(u_z-\frac{a^2}{b^2}u_z'\right)^2\right]^{3/2}}\right\}Rd\varphi \tag{23}$$

$$\mathcal{E}_y = \int_0^{2\pi}\left\{\frac{(u_y-u_y')}{\left[(u_x-u_x')^2+\left(u_y-u_y'\right)^2+(u_z-u_z')^2\right]^{3/2}} - \frac{\frac{a}{b}\left(u_y-\frac{a^2}{b^2}u_y'\right)}{\left[\left(u_x-\frac{a^2}{b^2}u_x'\right)^2+\left(u_y-\frac{a^2}{b^2}u_y'\right)^2+\left(u_z-\frac{a^2}{b^2}u_z'\right)^2\right]^{3/2}}\right\}Rd\varphi \tag{24}$$

$$\mathcal{E}_z = \int_0^{2\pi}\left\{\frac{(u_z-u_z')}{\left[(u_x-u_x')^2+\left(u_y-u_y'\right)^2+(u_z-u_z')^2\right]^{3/2}} - \frac{\frac{a}{b}\left(u_z-\frac{a^2}{b^2}u_z'\right)}{\left[\left(u_x-\frac{a^2}{b^2}u_x'\right)^2+\left(u_y-\frac{a^2}{b^2}u_y'\right)^2+\left(u_z-\frac{a^2}{b^2}u_z'\right)^2\right]^{3/2}}\right\}Rd\varphi \tag{25}$$

Once I had computed the magnetic fields using the above 18 equations, I modified the code to use the same 18 subroutines with similar denominators. The parameter $\frac{a}{b}$ was passed as an argument and both terms at the same $d\varphi$ step were added to the integral. The plots of these dimensionless electric and magnetic fields are seen below:

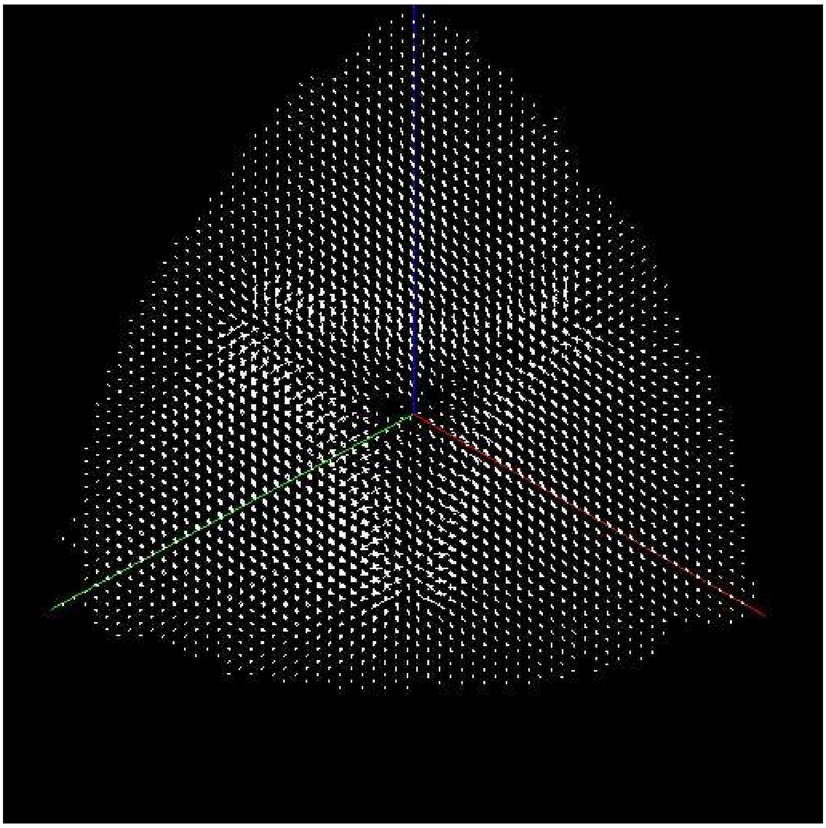

Figure 1. Electric field

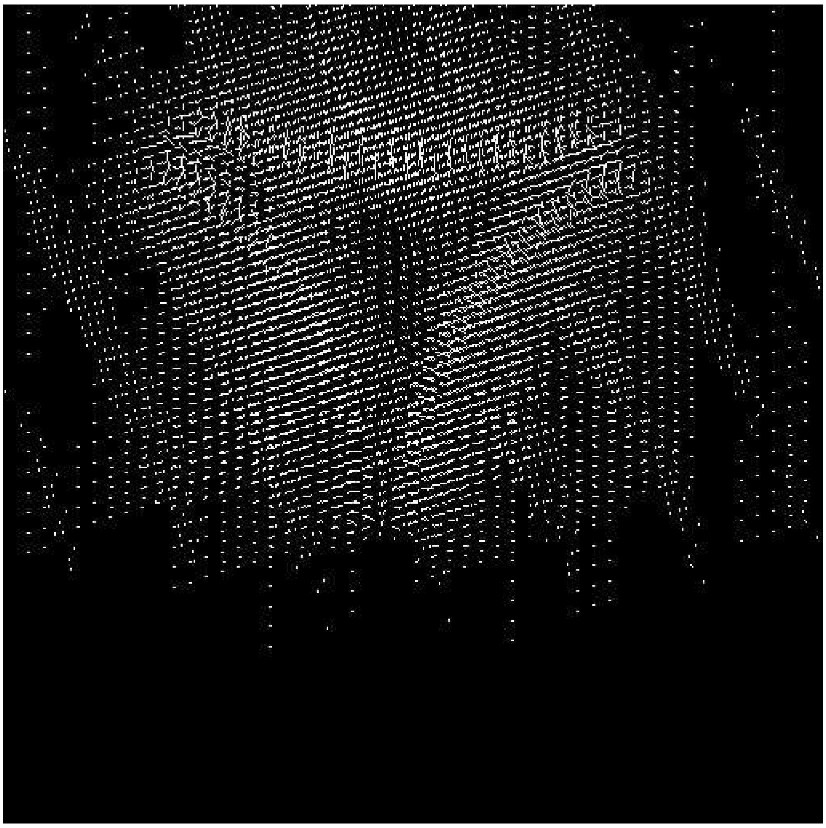

Figure 2. Magnetic field

To approach the physics of the polywell fusor one has a lot of approximations to choose from. I choose to use the basic plasma physics fluid model because it has a lot of history in electric and magnetic field environments which shows how useful the model can be. The basic equations are taken from "Principles of Plasma Physics", N.A. Krall and A.W. Trivelpiece, McGraw-Hill. In SI units we have

$$\frac{\partial f_e}{\partial t} + \vec{v}\cdot\vec{\nabla} f_e - \frac{e}{m}\left(\vec{E}_T + \vec{v}\times\vec{B}_T\right)\cdot\vec{\nabla}_v f_e = \frac{\partial f_e}{\partial t}|_c \tag{26}$$

$$\vec{\nabla}\cdot\vec{E}_T = -\frac{n_e e}{\varepsilon_0}\int f_e d\vec{v} + \frac{\rho_{\text{ext}}}{\varepsilon_0} \tag{27}$$

$$\vec{\nabla}\times\vec{B}_T = \mu_0\varepsilon_0\frac{\partial\vec{E}_T}{\partial t} - \mu_0 n_e e\int\vec{v} f_e d\vec{v} + \mu_0\vec{J}_{\text{ext}} \tag{28}$$

$$\vec{\nabla}\times\vec{E}_T = -\frac{\partial\vec{B}_T}{\partial t} \tag{29}$$

Here, f_e is the electron distribution function, $\vec{v}$ is the velocity vector, e is charge and m is the mass of an electron, $\vec{\nabla}_v$ is gradient with respect to velocity, n_e is the average electron density, ρ_{ext} is external electron distribution (not moving), $\vec{J}_{\text{ext}}$ is the moving external current density and $|_c$ refers to collisions. This last term will be ignored from here on out, but it should be noted this is where collisions enter the picture.

To make use of the external fields described above we can easily separate the total electric and magnetic fields into several components. The external fields are given by

$$\vec{\nabla} \cdot \vec{E}_{\text{ext}} = \frac{\rho_{\text{ext}}}{\varepsilon_0} \tag{30}$$

$$\vec{\nabla} \times \vec{E}_{\text{ext}} = -\frac{\partial \vec{B}_{\text{ext}}}{\partial t} \tag{31}$$

$$\vec{\nabla} \cdot \vec{B}_{\text{ext}} = 0 \tag{32}$$

$$\vec{\nabla} \times \vec{B}_{\text{ext}} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}_{\text{ext}}}{\partial t} + \mu_0 \vec{J}_{\text{ext}} \tag{33}$$

To start with, we take the fields in the coils as steady state and voltage on them as steady state as well so there is no electromagnetic coupling. The POPS design can include these terms later, and by writing everything fully it is easy to see where to put the fluctuating fields back in.

Using $\vec{E}$ and $\vec{B}$ as the fields created from the electrons it is easy to find the equations that show the connections between the electron fluid distribution and forces acting on it. subtracting the external equations from the particle distribution equations gives

$$\vec{\nabla} \cdot \vec{E} = -\frac{n_e e}{\varepsilon_0} \int f_e(\vec{x}, \vec{v}, t) d\vec{v} \tag{34}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} - \mu_0 n_e e \int \vec{v} f_e(\vec{x}, \vec{v}, t) d\vec{v} \tag{35}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{36}$$

If we assume (for the moment) that the fields created by the electrons is much weaker than the fields created by the coils, then we can make some very crude estimates of the behavior of the fluid. While it is obviously inaccurate for a real fusor, it does give some ideas on where to mount electron guns for injection and where *not* to mount electron guns as well.

With the weak field assumption the particle motion equation becomes

$$\frac{\partial f_e}{\partial t} + \vec{v} \cdot \vec{\nabla} f_e - \frac{e}{m} \left(\vec{E}_{\text{ext}} + \vec{v} \times \vec{B}_{\text{ext}} \right) \cdot \vec{\nabla}_v f_e = 0 \tag{37}$$

What I'd like to do now is transform this from a unit equation to a unitless equation to make a clear connection between the computed fields for the electrostatic and magnetostatic external fields already described. The units on the distrbution function are length^{-3} times velocity^{-3}. The units on $\vec{v}$ is velocity, $\vec{E}_{\text{ext}}$ is Volt/length, and the magnetic field is described in the first equation in this article (Tesla in SI units). To make this formula dimensionless, I multiply by (length^3 times velocity^3) along with writing the electric and magnetic fields in their dimensionless forms. I use f for the dimensionless particle distribution function, $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ for the dimensionless electric and magnetic fields, $\vec{\mathfrak{u}}$ for dimensionless velocity and $\vec{\mathfrak{r}}$ for dimensionless postion. The result of all this machination is

$$\frac{\partial f}{\partial t} = -\frac{v_0}{L} \vec{\mathfrak{u}} \cdot \frac{\partial f}{\partial \vec{\mathfrak{r}}} + \left(\frac{eV}{mLv_0} \vec{\mathcal{E}} + \frac{e\mu_0 I_0}{4\pi m L} \vec{\mathfrak{u}} \times \vec{\mathcal{B}} \right) \cdot \frac{\partial f}{\partial \vec{\mathfrak{u}}} \tag{38}$$

where v_0 is some arbitrary velocity which makes the terms dimensionless (inverse time actually, but I'll fix that in a minute). First I want to define the arbitrary velocity in terms of other variables which the problem has control over, then I'll work on moving terms around to a more convienient description.

The most obvious choice for a fundamental velocity is to assume that an electron which accelerates from dead still to the voltage on the grid has converted all its potential energy to kinetic energy. This gives

$$v_0 = \sqrt{\frac{2eV}{m}} \tag{39}$$

Putting this in for the arbitrary velocity and moving terms around we get

$$\frac{4\pi m L}{e\mu_0 I_0}\frac{\partial f}{\partial t} = -2\mathcal{C}_p \vec{\mathrm{u}} \cdot \frac{\partial f}{\partial \vec{\mathrm{r}}} + \left(\mathcal{C}_p \vec{\mathcal{E}} + \vec{\mathrm{u}} \times \vec{\mathcal{B}}\right) \cdot \frac{\partial f}{\partial \vec{\mathrm{u}}} \tag{40}$$

where

$$\mathcal{C}_p = \frac{2\pi}{\mu_0 I_0}\sqrt{\frac{2mV}{e}} \tag{41}$$

is what I call the "confinement parameter". This parameter is dimensionless so the whole equation can be scaled independent of real world constraints. Understanding the fluid distribution this way allows us to find reasonable combinations of voltages and currents which will create a workable fusion device, possibly of different sizes. It also helps to point out obvious combinations which simply can not work.

It is interesting that the time scale which makes the equation dimensionless is independent of the grid voltage. The time scale is given by the coefficient of the left hand side:

$$\text{time} = \frac{4\pi m L}{e\mu_0 I_0} \tag{42}$$

The Larmor frequency of an electron in a magnetic field is given by

$$\omega = \frac{eB}{m} \tag{43}$$

where B is the magnetic field in Tesla. Comparing these two equations we see that the fundamental magnetic field strength can be related to the Larmor frequency if we take

$$B = \frac{\mu_0 I_0}{2L} \tag{44}$$

The dimension of the device thus give us a fundamental scale for the Larmor frequency along with the current in the coil and the voltage on the grid. Plots of $\mathcal{C}_p\vec{\mathcal{E}} + \vec{\mathrm{u}} \times \vec{\mathcal{B}}$ for several values of $\mathcal{C}_p$ give us an idea of what the forces are on an electron fluid at various points in the polywell. Since this is a multidimensional space (3 dimensions for space, 3 for velocity and 1 (or two) for $\mathcal{C}_p$) it is non-trivial to get a feel for what is going on inside a polywell.

It should be clear that computing electron distributions over time is straight forward. Some assumptions on where to inject electrons in the first place need to be made using real physical devices, but a study of the "force volume" $\mathcal{C}_p\vec{\mathcal{E}} + \vec{\mathrm{u}} \times \vec{\mathcal{B}}$ will help to define the ideal electron gun locations.

It is clear that even simple models are fairly complicated. Only building a polywell device will tell us the real story. Models can help us find the best bet on what will work, and can certainly tell us what to avoid.

Part 2: Electron fluid static field

Let's continue the dimensionless approach with the rest of the electron fluid equations. Since I eliminated the subscript from $E_{\text{ext}} \longrightarrow \mathcal{E}$ in the above, let's transform the unsubscripted variables $E \longrightarrow \mathcal{E}_f$ to be subscripted. Inconsistent and insane, but for now I just want to see the math. When we write the book on polywell we can be clean about it!

$$\vec{\mathcal{E}}_f = \frac{L}{V}\vec{E} \tag{45}$$

$$\vec{\mathcal{B}}_f = \frac{4\pi L}{\mu_0 I_0}\vec{B} \tag{46}$$

And let us now define a new time variable which is dimensionless, based on the "time" equation above:

$$\nu = \frac{e\mu_0 I_0}{4\pi m L}t \tag{47}$$

With these substitutions, the plasma equation for the electron fluid becomes (with the assumption of no collisions)

$$\frac{\partial f}{\partial \nu} = -2\mathcal{C}_p \vec{\mathfrak{u}} \cdot \frac{\partial f}{\partial \vec{\mathfrak{r}}} + \left[\mathcal{C}_p\left(\vec{\mathcal{E}} + \vec{\mathcal{E}}_f\right) + \vec{\mathfrak{u}} \times \left(\vec{\mathcal{B}} + \vec{\mathcal{B}}_f\right)\right] \cdot \frac{\partial f}{\partial \vec{\mathfrak{u}}} \tag{48}$$

The electric field equation becomes

$$\frac{LV\varepsilon_0}{e}\frac{\partial}{\partial \vec{\mathfrak{r}}} \cdot \vec{\mathcal{E}}_f = -\tilde{n}\int f(\vec{\mathfrak{r}}, \vec{\mathfrak{u}}, \nu)\, d^3\mathfrak{u} \tag{49}$$

where $\tilde{n}$ is unitless number density. This parameter may be meaningless since $L^3 n = \tilde{n}$ is just a constant which could be incorporated into f. I leave it in the formulas just to keep track of n since we may need it when we add in Boron and proton ion densities.

Converting the magnetic field equation to dimensionless form we get

$$\frac{\partial}{\partial \vec{\mathfrak{r}}} \times \vec{\mathcal{B}}_f = \frac{\varepsilon_0 \mu_0 e V}{m}\frac{\partial \vec{\mathcal{E}}_f}{\partial \nu} - \frac{2e^2\mu_0}{mL}\mathcal{C}_p\tilde{n}\int \vec{\mathfrak{u}} f(\vec{\mathfrak{r}}, \vec{\mathfrak{u}}, \nu)\, d^3\mathfrak{u} \tag{50}$$

The coupling equation becomes

$$\frac{\partial}{\partial \vec{\mathfrak{r}}} \times \vec{\mathcal{E}}_f = -\frac{1}{2\mathcal{C}_p^2}\frac{\partial \vec{\mathcal{B}}_f}{\partial \nu} \tag{51}$$

At this point I have to make some kind of assumption about the fluid distribution. It requires a lot of computer effort to follow these equations in 3D, and even then some kind of initial distribution must be assumed. To see if the polywell concept makes any sense at all, let us assume an inital distribution of electrons is created in the well using a microwave generator. The form will be isotropic in both space and velocity and of course we pick a form which can be described easily in mathematics (otherwise it's impossible to proceed!)

$$f(\vec{\mathfrak{r}}, \vec{\mathfrak{u}}, \nu) = e^{-\alpha^2\mathfrak{u}^2}e^{-\beta^2\mathfrak{r}^2} \tag{52}$$

where we take $\mathfrak{u}^2 = \vec{\mathfrak{u}} \cdot \vec{\mathfrak{u}}$ and $\mathfrak{r}^2 = \vec{\mathfrak{r}} \cdot \vec{\mathfrak{r}}$. This distribution is almost realistic since the velocity term is just a Maxwellian distribution and the space term is a fast exponential decay. Some other form of space distribution may also be useful for analysis, but until a real device is built and measured this is really just fun.

Integrating over velocity for the electric field gives

$$\int f(\vec{\mathfrak{r}}, \vec{\mathfrak{u}}, \nu)\, d^3\mathfrak{u} = e^{-\beta^2\mathfrak{r}^2} \int_{-\infty}^{\infty} e^{-\alpha^2(\mathfrak{u}_x^2+\mathfrak{u}_y^2+\mathfrak{u}_z^2)} d\mathfrak{u}_x d\mathfrak{u}_y d\mathfrak{u}_z$$

$$= \left(\frac{\sqrt{\pi}}{\alpha}\right)^3 e^{-\beta^2\mathfrak{r}^2} \tag{53}$$

Plugging this into the gradiant of the electric field equation gives

$$\frac{LV\varepsilon_0}{e}\frac{\partial}{\partial\vec{\mathfrak{r}}} \cdot \vec{\mathcal{E}}_f = -\tilde{n}\left(\frac{\sqrt{\pi}}{\alpha}\right)^3 e^{-\beta^2\mathfrak{r}^2} \tag{54}$$

To just work on math and ignore physics for the moment, let's define

$$\vec{G} = \frac{LV\varepsilon_0}{e\tilde{n}}\frac{\alpha^3}{\pi^{3/2}}\vec{\mathcal{E}}_f \tag{55}$$

and the math problem to solve is just

$$\frac{\partial}{\partial\vec{\mathfrak{r}}} \cdot \vec{G} = e^{-\beta^2\mathfrak{r}^2} \tag{56}$$

Expanded this is

$$\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} = e^{-\beta^2(x^2+y^2+z^2)} \tag{57}$$

Where we can allow

$$G_x = G_x(x, y, z),\, G_y = G_y(x, y, z),\, G_z = G_z(x, y, z) \tag{58}$$

The first thing to do is make the assumption that

$$G_j = A_j(x, y, z)e^{-\beta^2(x^2+y^2+z^2)} \tag{59}$$

which when plugged into the gradient equation gives

$$\frac{\partial A_x}{\partial x} - 2A_x\beta^2 x + \frac{\partial A_y}{\partial y} - 2A_y\beta^2 y + \frac{\partial A_z}{\partial z} - 2A_z\beta^2 z - 1 = 0 \tag{60}$$

To proceed, I assume that the A_j are independent of each other, i.e.

$$A_x = A_x(x),\, A_y = A_y(y),\, A_z = A_z(z) \tag{61}$$

This is probably not a valid assumption for the real world, but it will tell us immediatly if the polywell is impossible. The following is proof only that there are more details to look at carefully.

With the assumption that the A_j's are independent we can also assume they are isotropic and equivelent so we have 3 equations of the form

$$\frac{\partial A_j}{\partial \mathfrak{r}_j} - 2A_j\beta^2\mathfrak{r}_j - \frac{1}{3} = 0 \tag{62}$$

This is straight forward to solve using an integration factor (see "Advanced Calculus for Applications" by F. B. Hildebrand or similar texts) and we get

$$A_j = e^{\beta^2\mathfrak{r}_j^2}\left[\frac{-1}{3}\int e^{-\beta^2 t^2} dt + C_j\right] \tag{63}$$

From Gradshteyn & Ryzhik 3.321.2 we have

$$\int_0^u e^{-q^2x^2}dx = \frac{\sqrt{\pi}}{2q}\mathrm{erf}(qx) \tag{64}$$

where erf(x) is the error function (see "Handbook of Mathematical Functions" by Abramowitz and Stegun for details).

Then the solution to the electric field is

$$\vec{G} = \frac{LV\varepsilon_0}{e\,\tilde{n}}\frac{\alpha^3}{\pi^{3/2}}\vec{\mathcal{E}}_f = \frac{-1}{3\beta}\Big\{[\mathrm{erf}(\beta x) + C_x]e^{-\beta^2(y^2+z^2)}\hat{x}$$

$$+ [\mathrm{erf}(\beta y) + C_y]e^{-\beta^2(x^2+z^2)}\hat{y} + [\mathrm{erf}(\beta z) + C_z]e^{-\beta^2(x^2+y^2)}\hat{z}\Big\} \tag{65}$$

What does this humongous mess really tell us? Well, for starters, it says that the electric field caused by the electron fluid goes to zero at the center of the well and becomes more negative as we go outward. It is proportional to the spatial distribution of the fluid near its outer edges. The constants of integration have to be determined from other assumptions, but since this whole excersize is about theoretical considerations I'll ignore them for the moment (set them equal to zero).

Let's put the assumed electron fluid distribution function into the plasma equation and see what happens:

$$\frac{\partial f}{\partial \nu} = \left[4\mathcal{C}_p\beta^2\vec{\mathfrak{u}}\cdot\vec{\mathfrak{r}} - 2\alpha^2\mathcal{C}_p\left(\vec{\mathcal{E}} + \vec{\mathcal{E}}_f\right)\cdot\vec{\mathfrak{u}}\right]f \tag{66}$$

The important thing to notice here is that $\vec{\mathcal{B}}$ drops out! The isotropic velocity distribution means the magnetic field does not participate in the fluid motion. This is counter intuitive because it is clear the electron motion IS determined by the magnetic field.

What this really means is that simple mathematical assumptions can only take us so far. The conditions for stability are obvious if we do a simple separation of variables argument and divide both sides of the above by f and multiply by $\partial\nu$:

$$f = \exp(2\mathcal{C}_p\vec{\mathfrak{u}}\cdot\int\left[2\beta^2\vec{\mathfrak{r}} - \alpha^2\left(\vec{\mathcal{E}} + \vec{\mathcal{E}}_f\right)\right]d\nu \tag{67}$$

The thermal properties times the electric field must be greater than the spatial properties. Otherwise, the fluid explodes. The electric field pressure has to be large enough, and the electrons themselves decrease this field strength as we saw above. This is a zeroth order requirement - if the thermal temperature of the electons is low then the spatial distribution will not be a problem (large α means low temperature). The physical size determines what that temperature can be.

With the theoretical description we can explore further. But real polywell parameters can be plugged into the above theory and we can compare what works with what doesn't to see how useful the above theory and its associated assumptions really are.

Part 3: Single Electron Orbit

To find out if the Polywell is stable for even a single electron involves solving the fundamental Newtonian physics of $F = ma$, but in 3 dimensions. As shown in part 1, we already have a set of fixed external electric and magnetic fields. For the following we will ignore the electron's self field and radiation as well as relativistic corrections. While simple, it is still a messy problem analytically.

In full three dimensional notation, we have

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = -e(\vec{E} + \vec{v} \times \vec{B}) \tag{68}$$

where $-e$ is the charge on an electron, m is the electron mass and $\vec{v}$ is the electron's velocity. Expanding this in all its glory gives

$$m\frac{dv_x}{dt} = -e(E_x + v_y B_z - v_z B_y) \tag{69}$$

$$m\frac{dv_y}{dt} = -e(E_y + v_z B_x - v_x B_z) \tag{70}$$

$$m\frac{dv_z}{dt} = -e(E_z + v_x B_y - v_y B_x) \tag{71}$$

This is a system of differential equations, and it is only of first order. In the following I apply the methods described in F.B. Hildebrand "Advanced Calculus for Applications" (chapter 1). One of his notations is to replace $\frac{d}{dt}$ with D but I wish to move into unitless form (as in sections 1 and 2 above) so I'll start with $\frac{d}{dt} = D'$. Rewriting the above in the linear equations format gives

$$\frac{m}{e}D'v_x + B_z v_y - B_y v_z = -E_x \tag{72}$$

$$-B_z v_x + \frac{m}{e}D'v_y + B_x v_z = -E_y \tag{73}$$

$$B_y v_x - B_x v_y + \frac{m}{e}D'v_z = -E_z \tag{74}$$

Using the same transformations as in equations 39, 41, 45, 46 and 47 the above (72 - 74) become

$$D\mathfrak{u}_x + \mathcal{B}_z\mathfrak{u}_y - \mathcal{B}_y\mathfrak{u}_z = -\mathcal{C}_p\mathcal{E}_x \tag{75}$$

$$-\mathcal{B}_z\mathfrak{u}_x + D\mathfrak{u}_y + \mathcal{B}_x\mathfrak{u}_z = -\mathcal{C}_p\mathcal{E}_y \tag{76}$$

$$\mathcal{B}_y\mathfrak{u}_x - \mathcal{B}_x\mathfrak{u}_y + D\mathfrak{u}_z = -\mathcal{C}_p\mathcal{E}_z \tag{77}$$

where $D = \frac{d}{d\nu}$ (see 47 for definition of ν). As in Hildebrand, the solution for the homogeneous portion is found from

$$\Delta = \begin{vmatrix} D & \mathcal{B}_z & -\mathcal{B}_y \\ -\mathcal{B}_z & D & \mathcal{B}_x \\ \mathcal{B}_y & -\mathcal{B}_x & D \end{vmatrix} \tag{78}$$

$$= D^3 + (\mathcal{B}_x^2 + \mathcal{B}_y^2 + \mathcal{B}_z^2)D = 0 \tag{79}$$

The homogeneous solutions are of the form

$$\mathfrak{u}_h = e^{r\nu} \tag{80}$$

so $D\mathfrak{u}_h = re^{r\nu}$ and 79 becomes

$$r(r^2 + \mathcal{B}_x^2 + \mathcal{B}_y^2 + \mathcal{B}_z^2) = 0 \tag{81}$$

which has three solutions

$$r=0,\ r=\pm i\sqrt{\mathcal{B}_x^2+\mathcal{B}_y^2+\mathcal{B}_z^2} \tag{82}$$

The particluar solution for $\mathfrak{u}_x$ is found by setting Δ equal to

$$\Delta \mathfrak{u}_x=\begin{vmatrix} -\mathcal{C}_p\mathcal{E}_x & \mathcal{B}_z & -\mathcal{B}_y \\ -\mathcal{C}_p\mathcal{E}_y & D & \mathcal{B}_x \\ -\mathcal{C}_p\mathcal{E}_z & -\mathcal{B}_x & D \end{vmatrix} \tag{83}$$

$$=-\mathcal{C}_p\mathcal{E}_x D^2+\mathcal{C}_p(\mathcal{E}_y\mathcal{B}_z-\mathcal{E}_z\mathcal{B}_y)D-\mathcal{C}_p\mathcal{B}_x(\mathcal{E}_x\mathcal{B}_x+\mathcal{E}_y\mathcal{B}_y+\mathcal{E}_z\mathcal{B}_z) \tag{84}$$

Setting 78 equal to 83, substituting $\mathfrak{u}_x=e^{r_l\nu}$, and grinding gives

$$r_l^3+\mathcal{C}_p\mathcal{E}_x r_l^2+r_l\left[\mathcal{B}_x^2+\mathcal{B}_y^2+\mathcal{B}_z^2+\mathcal{C}_p(\mathcal{E}_y\mathcal{B}_z-\mathcal{E}_z\mathcal{B}_y)\right]+\mathcal{C}_p\mathcal{B}_x(\mathcal{E}_x\mathcal{B}_x+\mathcal{E}_y\mathcal{B}_y+\mathcal{E}_z\mathcal{B}_z)=0 \tag{85}$$

To shorten the notation, let's write

$$\vec{\mathcal{B}}\cdot\vec{\mathcal{B}}=\mathcal{B}_x^2+\mathcal{B}_y^2+\mathcal{B}_z^2 \tag{86}$$

and

$$\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}=\mathcal{E}_x\mathcal{B}_x+\mathcal{E}_y\mathcal{B}_y+\mathcal{E}_z\mathcal{B}_z \tag{87}$$

Using the solution to cubic equations found in CRC Handbook "Standard Mathematical Tables" 21st edition page 103, we can take

$$s=r_l-\frac{\mathcal{C}_p\mathcal{E}_x}{3} \tag{88}$$

to transform 85 to the form

$$s^3+as+b=0 \tag{89}$$

where

$$a=\vec{\mathcal{B}}\cdot\vec{\mathcal{B}}+\mathcal{C}_p(\mathcal{E}_z\mathcal{B}_y-\mathcal{E}_y\mathcal{B}_z)-\frac{\mathcal{C}_p^2\mathcal{E}_x^2}{3} \tag{90}$$

and

$$b=2\frac{\mathcal{C}_p^3\mathcal{E}_x^3}{27}-\frac{\mathcal{C}_p\mathcal{E}_x}{3}\left[\vec{\mathcal{B}}\cdot\vec{\mathcal{B}}+\mathcal{C}_p(\mathcal{E}_z\mathcal{B}_y-\mathcal{E}_y\mathcal{B}_z)\right]+\mathcal{C}_p\mathcal{B}_x\left(\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}\right) \tag{91}$$

Notice that there are some common terms in 90 and 91. Let's simplify the notation some more and write

$$\alpha_x=\vec{\mathcal{B}}\cdot\vec{\mathcal{B}}+\mathcal{C}_p(\mathcal{E}_z\mathcal{B}_y-\mathcal{E}_y\mathcal{B}_z) \tag{92}$$

$$\gamma_x=\mathcal{C}_p\mathcal{E}_x \tag{93}$$

$$\delta_x=\frac{\mathcal{B}_x}{\mathcal{E}_x}\left(\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}\right) \tag{94}$$

and I use the subscript x to signify that this particular solution is for $\mathfrak{u}_x$. The solutions for s are give with the variables

$$A=\sqrt[3]{-\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}},\ B=\sqrt[3]{-\frac{b}{2}-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}} \tag{95}$$

and solutions s are

$$s = A + B, -\frac{A+B}{2} + \sqrt{3}\frac{A-B}{2}, -\frac{A+B}{2} - \sqrt{3}\frac{A-B}{2} \tag{96}$$

Substiting 92-94 into 90 and 91 then grinding through the algebra for the insides of the cube root of 95 gives the following mess

$$-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} = -\frac{\gamma_x}{54}(2\gamma_x^2 - 9\alpha_x + 27\delta_x) \pm \frac{1}{\sqrt{3}}\left[\left(\frac{\alpha^2}{9} - \gamma_x^2\delta_x\right)\left(\alpha_x - \frac{\gamma_x^2}{4}\right) + \frac{\gamma_x^2\delta_x}{4}\left(\frac{5}{9}\gamma_x^2 - 3\delta_x\right)\right]^{1/2} \tag{97}$$

If the term inside the square root in 97 is negative, we have a complex number representation. We can easily take the cube root of this going into polar notation and dividing the angle by 3 as well as taking the cube root of the magnitude. If we take the general form of 97 to be $x + iy$ then the cube root is given by

$$x' + iy' = \left(x^2 + y^2\right)^{1/6}\cos\left(\frac{1}{3}\tan^{-1}\frac{y}{x}\right) + i\left(x^2 + y^2\right)^{1/6}\sin\left(\frac{1}{3}\tan^{-1}\frac{y}{x}\right) \tag{98}$$

(I'm running out of variables, the x, x', y, y' here are dummy variables which represent a *form*, the actual variables will be really messy!)

Note that A and B will be complex congugates if $\frac{b^2}{4} + \frac{a^3}{27} < 0$. This leads to $A + B$ being the real part and $A - B$ being the imaginary part. To find the final solution, we must unwind our way back through the definitions of r_l in 88 from the solutions s in 96 which are determined by putting 97 into 98.

The trick here is that we have solutions for every point in space. The same formula found setting 80 equal to 83 must be done for $\Delta\mathfrak{u}_y$ and $\Delta\mathfrak{u}_z$. Fortunatly, the system is really symmetric, and all we really have to do is change subscripts in 92 - 94 by sending $x \to y$, $y \to z$, $z \to x$. This makes writing subroutines really easy. The same basic form applies to all dimensions.

The imaginary exponential solutions need to be combined to get the sine and cosine functions out so our computer will have a much easier time dealing with things. There is really a lot of exception handling here. The basic process will be to start with a particle in a position similar to either the center of the polywell or the electron source points of WB-6. Assume it has zero velocity to start, and see where it goes by computing the solution to the equations 92, 93, 94 based on the location, then finding the r_l homogeneous and particular in all three dimensions, and find the exponents.

Since the equations change as a function of postion, and the position changes with time, we have to take fairly small steps in time and recompute all the solutions to the differential equations. Since we already have the fields, this is "straight forward". The basic solution will be of the form

$$\mathfrak{u}_j \sim a_j + b_j r_+ + c_j r_- + d_j r_{p0} + e_j r_{p1} + f_j r_{p2} \tag{99}$$

where the first 3 terms are the homogeneous solution and the last three will be particular to that index j. The particle position is then simply

$$\mathfrak{r}_j = \mathfrak{u}_j \, \Delta\nu \tag{100}$$

Clearly this is a non-trival task, but it will tell for certain if the polywell has stable orbits for even a low density electron current. Assuming these orbits can be shown, finding a fluid solution from section 2 will be easier because we'll have an idea of how a non-interaction current should behave.