Electron Fluid in a Polywell

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An electron fluid can be thought of as a distribution of particles in space and velocity as shown in the "Virtual Polywell" write up. The fundamental equations I want to explore here are given by 48 and 49 in that report. These are:

$$\frac{\partial f}{\partial \nu} = -2\mathcal{C}_p \vec{\mathfrak{u}} \cdot \frac{\partial f}{\partial \vec{\mathfrak{r}}} + \left[\mathcal{C}_p\left(\vec{\mathcal{E}} + \vec{\mathcal{E}}_f\right) + \vec{\mathfrak{u}} \times \left(\vec{\mathcal{B}} + \vec{\mathcal{B}}_f\right)\right] \cdot \frac{\partial f}{\partial \vec{\mathfrak{u}}} \tag{1}$$

$$\frac{LV\varepsilon_0}{e}\frac{\partial}{\partial \vec{\mathfrak{r}}} \cdot \vec{\mathcal{E}}_f = -\tilde{n}\int f(\vec{\mathfrak{r}}, \vec{\mathfrak{u}}, \nu)\, d^3\mathfrak{u} \tag{2}$$

From Krall & Trivelpiece "Principles of Plasma Physics" (pg 355) we find

$$\phi_f(\vec{\mathfrak{r}}, t) = \sum_\alpha q_\alpha \int \frac{f_\alpha(\vec{\mathfrak{r}}\,', \vec{\mathfrak{u}}\,', t)}{|\vec{\mathfrak{r}} - \vec{\mathfrak{r}}\,'|} d\vec{\mathfrak{r}}\,' \, d\vec{\mathfrak{u}}\,' \tag{3}$$

(note: cgs units!) Since we can use

$$\frac{\partial}{\partial \vec{\mathfrak{r}}}\phi_f = \vec{\mathcal{E}}_f \tag{4}$$

we can compute the $\vec{\mathcal{E}}_f$ from the potential rather than from equation (2). This allows us to find the force on the fluid in equation (1) from an alternate direction. The reason for this is so we can include the potential from the MaGrid directly in the calculation of the particle distribution.

I only want to look at $\nu = 0$, the initial conditions. The distribution

$$f(\vec{\mathfrak{r}}, \vec{\mathfrak{u}}, 0) = \tilde{R}(\vec{\mathfrak{r}})\tilde{\mathfrak{U}}(\vec{\mathfrak{u}}) \tag{5}$$

can be assumed to be of any form we like. In one of Bussard's papers he describes the plasma density as falling off as $\frac{1}{r^2}$ so for a radial function I chose the following form

$$\tilde{R}(\vec{\mathfrak{r}}) = \frac{R'(\alpha, \beta, s)}{e^{-\alpha\mathfrak{r}^2} + \beta\mathfrak{r}^2} \tag{6}$$

where α and β are arbitrary parameters and s is the radius of the outside sphere in units of L, the distance from the center of the Polywell to the center of the coils and $\mathfrak{r}^2 = \vec{\mathfrak{r}} \cdot \vec{\mathfrak{r}}$. The advantage of this form is that it goes to 1 as $\mathfrak{r} \to 0$. For $\alpha > \beta$ the distribution (4) is humped similar to the second well shown in some Polywell papers. The hump happens when

$$\mathfrak{r} = \sqrt{\frac{1}{\alpha}\ln\frac{\alpha}{\beta}} \tag{7}$$

$R'(\alpha,\beta,s)$ is simply a normalization factor:

$$R'(\alpha,\beta,s)=\left[2\pi\int_0^s\frac{r^2dr}{e^{-\alpha r^2}+\beta r^2}\right]^{-1} \tag{8}$$

In the velocity dimension I take the electron distribution as Maxwellian every where. This is a reasonable assumption for initial conditions, especially if the plasma is created with microwaves. For this initial conditions estimate, it is a very simple condition which allows the theory to proceed with a lot less effort.

In addition to the Maxwellian velocity, I take the potential which the electrons find themselves in as part of the distribution. This makes sense since more electrons will be at a place of high positive potential and very few will be in a place of negative potential. The velocity form of the distribution is

$$\tilde{\mathfrak{U}}(\vec{\mathfrak{u}})=\mathfrak{u}'e^{-\frac{q\Phi+\frac{m}{2}(v_0\vec{\mathfrak{u}}\cdot v_0\vec{\mathfrak{u}})}{\bar{E}}} \tag{9}$$

Where q is the particle charge, Φ is the potential at some point in space, m is the particle mass, and $v_0\vec{\mathfrak{u}}$ is the particle velocity. The value $\bar{E}$ is the average electron energy or equivalently, the temperature. Form (7) actually makes $\tilde{\mathfrak{U}}(\vec{\mathfrak{u}})$ a function of space as well as velocity. I am going to assume that the particle distribution is Maxwellian everywhere, so Φ does not depend on velocity. Reality does not work this way, but the formulas would be intractable if this assumption is not made.

To convert equation (7) to unitless form, let's take the average energy to be a function of the voltage on the MaGrid:

$$\bar{E}=geV \tag{10}$$

where g is now related to the average energy as a function of the MaGrid voltage. I consider this a knob in the model for seeing how temperature affects the particle distribution.

As done in the Virtual Polywell document, I take

$$v_0=\sqrt{\frac{2eV}{m}} \tag{11}$$

since the charge on the electrons is negative $q=-e$. This allows me to convert (7) to dimensionless form as

$$\tilde{\mathfrak{U}}(\vec{\mathfrak{u}})=\mathfrak{u}'e^{-\frac{-e\Phi+eV\vec{\mathfrak{u}}\cdot\vec{\mathfrak{u}}}{geV}} \tag{12}$$

which I can write as

$$\tilde{\mathfrak{U}}(\vec{\mathfrak{u}})=\mathfrak{u}'e^{\frac{\phi-\vec{\mathfrak{u}}\cdot\vec{\mathfrak{u}}}{g}} \tag{13}$$

Since ϕ is only a function of position and not velocity, we can find $\mathfrak{u}'$ as the normalization constant using

$$\mathfrak{u}'=\left[\int_{-\infty}^{\infty}e^{-\frac{\vec{\mathfrak{u}}\cdot\vec{\mathfrak{u}}}{g}}d^3\mathfrak{u}\right]^{-1}=\frac{1}{(\pi g)^{3/2}} \tag{14}$$

We can relate the charge on the MaGrid to the voltage we set it at using the same relation after equation (22):

$$dq=4\pi\varepsilon_0Vdl \tag{15}$$

This relationship can be used to compute the potential everywhere in the volume based on the same form of the potential shown in equation (22) of the Virtual Polywell paper. Every ring of the MaGrid has a matching image charge. The formulas for the potential from each ring are given here:

$$\psi_{+x} = R\int_0^{2\pi} \frac{d\varphi}{\left[(u_x-1)^2+(u_y-R\cos\varphi)^2+(u_z-R\sin\varphi)^2\right]^{1/2}} \tag{16}$$

$$-R\int_0^{2\pi} \frac{s}{\sqrt{1+R^2}} \frac{d\varphi}{\left[\left(u_x-\frac{s^2}{1+R^2}\right)^2+\left(u_y-\frac{s^2}{1+R^2}R\cos\varphi\right)^2+\left(u_z-\frac{s^2}{1+R^2}R\sin\varphi\right)^2\right]^{\frac{1}{2}}}$$

$$\psi_{-x} = R\int_0^{2\pi} \frac{d\varphi}{\left[(u_x+1)^2+(u_y-R\cos\varphi)^2+(u_z-R\sin\varphi)^2\right]^{1/2}} \tag{17}$$

$$-R\int_0^{2\pi} \frac{s}{\sqrt{1+R^2}} \frac{d\varphi}{\left[\left(u_x+\frac{s^2}{1+R^2}\right)^2+\left(u_y-\frac{s^2}{1+R^2}R\cos\varphi\right)^2+\left(u_z-\frac{s^2}{1+R^2}R\sin\varphi\right)^2\right]^{\frac{1}{2}}}$$

$$\psi_{+y} = R\int_0^{2\pi} \frac{d\varphi}{\left[(u_x+R\cos\varphi)^2+(u_y-1)^2+(u_z-R\sin\varphi)^2\right]^{1/2}} \tag{18}$$

$$-R\int_0^{2\pi} \frac{s}{\sqrt{1+R^2}} \frac{d\varphi}{\left[\left(u_x+\frac{s^2}{1+R^2}R\cos\varphi\right)^2+\left(u_y-\frac{s^2}{1+R^2}\right)^2+\left(u_z-\frac{s^2}{1+R^2}R\sin\varphi\right)^2\right]^{\frac{1}{2}}}$$

$$\psi_{-y} = R\int_0^{2\pi} \frac{d\varphi}{\left[(u_x+R\cos\varphi)^2+(u_y+1)^2+(u_z-R\sin\varphi)^2\right]^{1/2}} \tag{19}$$

$$-R\int_0^{2\pi} \frac{s}{\sqrt{1+R^2}} \frac{d\varphi}{\left[\left(u_x+\frac{s^2}{1+R^2}R\cos\varphi\right)^2+\left(u_y+\frac{s^2}{1+R^2}\right)^2+\left(u_z-\frac{s^2}{1+R^2}R\sin\varphi\right)^2\right]^{\frac{1}{2}}}$$

$$\psi_{+z} = R\int_0^{2\pi} \frac{d\varphi}{\left[(u_x-R\cos\varphi)^2+(u_y-R\sin\varphi)^2+(u_z-1)^2\right]^{1/2}} \tag{20}$$

$$-R\int_0^{2\pi} \frac{s}{\sqrt{1+R^2}} \frac{d\varphi}{\left[\left(u_x-\frac{s^2}{1+R^2}R\cos\varphi\right)^2+\left(u_y-\frac{s^2}{1+R^2}R\sin\varphi\right)^2+\left(u_z-\frac{s^2}{1+R^2}\right)^2\right]^{\frac{1}{2}}}$$

$$\psi_{-z} = R\int_0^{2\pi} \frac{d\varphi}{\left[(u_x-R\cos\varphi)^2+(u_y-R\sin\varphi)^2+(u_z+1)^2\right]^{1/2}} \tag{21}$$

$$-R\int_0^{2\pi} \frac{s}{\sqrt{1+R^2}} \frac{d\varphi}{\left[\left(u_x-\frac{s^2}{1+R^2}R\cos\varphi\right)^2+\left(u_y-\frac{s^2}{1+R^2}R\sin\varphi\right)^2+\left(u_z+\frac{s^2}{1+R^2}\right)^2\right]^{\frac{1}{2}}}$$

The total potential due to just the MaGrid everywhere in the sphere is then

$$\psi = \psi_{+x} + \psi_{-x} + \psi_{+y} + \psi_{-y} + \psi_{+z} + \psi_{-z} \tag{22}$$

and I take the potential everywhere in the sphere as

$$\phi = \phi_f(u_x, u_y, u_z) + \psi(u_x, u_y, u_z) \tag{23}$$

In this paper I'm only considering one species of particles - electrons. So the formula (3) can be modified to drop the sum over species. I also include the $\tilde{n}$ as a pure number - the total number of electrons in the sphere. Looking only at $t = 0$ and putting (6), (13) and (14) into (5) gives

$$f(\vec{\mathfrak{r}}, \vec{\mathfrak{u}}, 0) = \frac{R'(\alpha, \beta, s)}{e^{-\alpha\mathfrak{r}^2} + \beta\mathfrak{r}^2} \frac{1}{(\pi g)^{3/2}} e^{\frac{\phi - \vec{\mathfrak{u}} \cdot \vec{\mathfrak{u}}}{g}} \tag{24}$$

Putting (24) into (3) gives

$$\phi_f(\vec{\mathfrak{r}}, t) = -\frac{e\,\tilde{n}}{4\pi\varepsilon_0 VL} \int \frac{d\vec{\mathfrak{r}}'\, d\vec{\mathfrak{u}}'}{|\vec{\mathfrak{r}} - \vec{\mathfrak{r}}'|} \frac{R'(\alpha, \beta, s)}{e^{-\alpha\mathfrak{r}'^2} + \beta\mathfrak{r}'^2} \frac{1}{(\pi g)^{3/2}} e^{\frac{\phi - \vec{\mathfrak{u}}' \cdot \vec{\mathfrak{u}}'}{g}} \tag{25}$$

Integrating (25) over all velocity we get the following as the initial distribution and its potential everywhere in the sphere:

$$\phi_f(u_x, u_y, u_z) = -\frac{e\,\tilde{n}}{4\pi\varepsilon_0 VL} \iiint \frac{du_x' du_y' du_z'}{\left[(u_x - u_x')^2 + (u_y - u_y')^2 + (u_z - u_z')^2\right]^{1/2}} \times \tag{26}$$

$$\frac{R'(\alpha, \beta, s)}{e^{-\alpha\mathfrak{r}'^2} + \beta\mathfrak{r}'^2} e^{-\frac{1}{g}\left(\psi + \phi_f\right)}$$

and I've taken

$$\mathfrak{r}^2 = u_x^2 + u_y^2 + u_z^2 \tag{27}$$

So what does this really mean??

What it says is that the electron fluid seen as a particle distribution which is Maxwellian has a potential which is everywhere dependent on both the grid potential and itself. To find the initial conditions requires solving equation (26) for the assumed distribution of particles so we can determine the forces acting on all the particles.

In a sense this is not very realistic since the particles can not just magically appear. However, it is clear that at any time there is any particle distribution we have the same problem as seen from equation (3). The potential everywhere in the sphere depends on all the particles in the sphere, so the potential at any point in space depends on how the particles are distributed in that space.

Since the formulas for ψ were computed in spherical coordinates and tabulated that way to conserve memory, I will compute (26) in spherical coordinates as well. The transformation from Cartesian to spherical is given by the following:

$$u_x = r\cos\theta\cos\varphi$$

$$u_y = r\cos\theta\sin\varphi \tag{28}$$

$$u_z = r\sin\theta$$

Note that I take θ up from the x, y plane rather than down from the z axis. This was convienient for the calculation of $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ fields. It is straight forward to show that

$$\left|\vec{\mathfrak{r}} - \vec{\mathfrak{r}}'\right| = \sqrt{r^2 + r'^2 - rr'[\cos(\theta - \theta')(1 + \cos(\varphi - \varphi')) - \cos(\theta + \theta')(1 - \cos(\varphi - \varphi'))]} \tag{29}$$

There are obviously many other trigonometric forms one could use but this has the sum and difference of all the angles built into the form. The differential portions become

$$du'_x du'_y du'_z = r'^2 \cos\theta dr' \, d\theta' \, d\varphi' \tag{30}$$

So (26) transforms to spherical coordinates as

$$\phi_f(r, \theta, \varphi) = -\frac{e\,\tilde{n}}{4\pi\varepsilon_0 VL} \int\!\!\int\!\!\int \frac{R'(\alpha, \beta, s)}{e^{-\alpha\mathfrak{r}'^2} + \beta\mathfrak{r}'^2} e^{-\frac{1}{g}\left(\psi + \phi_f\right)}$$
$$\frac{r'^2 \cos\theta dr' \, d\theta' \, d\varphi'}{\sqrt{r^2 + r'^2 - rr'[\cos(\theta - \theta')(1 + \cos(\varphi - \varphi')) - \cos(\theta + \theta')(1 - \cos(\varphi - \varphi'))]}} \tag{31}$$

and the integration over the volume is done with $r' = 0 \to s, \theta' = -\frac{\pi}{2} \to \frac{\pi}{2}, \varphi = 0 \to 2\pi$. The value $\frac{e\,\tilde{n}}{4\pi\varepsilon_0 VL}$ is a dimensionless parameter and should be treated as knob just as the parameter g is used as knob to get an idea of how the potential behaves as a function of particle density, MaGrid voltage and electron temperature. The lookup table for $\psi(r, \theta, \varphi)$ is easy to compute from (16) through (22) as a one part in 48 of the entire sphere. It is simply a matter of book keeping to include the entire volume to properly compute (31). Also note that $R'(\alpha, \beta, s)$ only needs to be computed once for any particluar distribution.